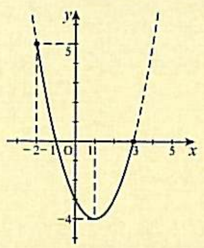
Aim For Program Completion!

K (Pre Calculus/Math Analysis)

Quadratic Functions



$y = x^2 - 2x - 3 \quad (-2 \leq x \leq 3)$

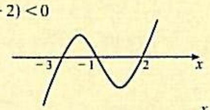
$y = (x-1)^2 - 4$

Maximum value: 5, at $x = -2$
Minimum value: -4, at $x = 1$

Range: $-4 \leq y \leq 5$

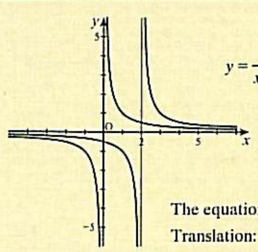
Quadratic and Higher Degree Inequalities

$(x+3)(x+1)(x-2) < 0$



$x < -3,\ -1 < x < 2$

Fractional Functions

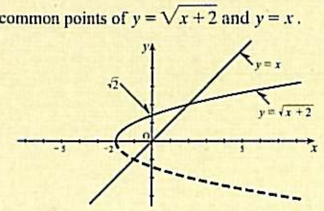


$y = \frac{1}{x-2}$

The equations of the asymptotes: $x=2, y=0$
Translation: 2 unit(s) along the x-axis

Irrational Functions

Find the common points of $y = \sqrt{x+2}$ and $y = x$.



[Sol] Let $\sqrt{x+2} = x$

$$x + 2 = x^2$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1, 2$$

From the graph, $x = -1$ is an extraneous solution.
Therefore the common point is (2, 2).

Exponential Functions

- $\left(5^{\frac{1}{2}} + 5^{-\frac{1}{2}}\right)\left(5^{\frac{1}{2}} - 5^{-\frac{1}{2}}\right) = \left(5^{\frac{1}{2}}\right)^2 - \left(5^{-\frac{1}{2}}\right)^2 = 5 - \frac{1}{5} = \frac{24}{5}$
- $4^x = 8$
  $2^{2x} = 2^3$
  $2x = 3$
  $x = \frac{3}{2}$

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L (Calculus)

Logarithmic Functions

- $\log_4 8 = x$
  $4^x = 8$
  $2^{2x} = 2^3$
  $x = \frac{3}{2}$
- $\log_3 27 + \log_3 18 - \log_3 2$
  $= \log_3 3^3 + \log_3(2 \cdot 3^2) - \log_3 2$
  $= 3 + (\log_3 2 + 2) - \log_3 2$
  $= 5$

Modulus Functions

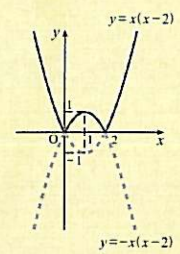
$y = |x(x-2)|$

When $x(x-2) \geq 0$

When $x \leq 0$ or $2 \leq x$,
$y = x(x-2)$

When $x(x-2) < 0$

When $0 < x < 2$,
$y = -x(x-2)$



Differentiation

- $\lim_{x \to -2} \frac{x^2 - x - 6}{x+2} = \lim_{x \to -2} \frac{(x+2)(x-3)}{(x+2)} = \lim_{x \to -2}(x-3) = -5$
- $y = 2x^3 \quad y' = 6x^2$
- Find the relative extreme values of $y = x^3 - 6x^2 + 9x - 4$.

$y' = 3x^2 - 12x + 9$
$= 3(x^2 - 4x + 3)$
$= 3(x-1)(x-3) = 0$
$x = 1, 3$

x	...	1	...	3	...
y'	+	0	−	0	+
y	↗	0	↘	−4	↗

The relative maximum value is 0, at $x = 1$.

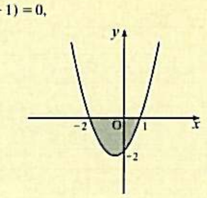
The relative minimum value is -4, at $x = 3$.

Integration

- $\int 6x^2 dx = 2x^3 + C$
- Find the area, S, of the region enclosed by $y = x^2 + x - 2$ and the x-axis.

[Sol] From $y = x^2 + x - 2 = (x+2)(x-1) = 0$,
$x = -2, 1$

$$S = -\int_{-2}^{1}(x^2 + x - 2)dx$$
$$= -\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x\right]_{-2}^{1} = \frac{9}{2}$$



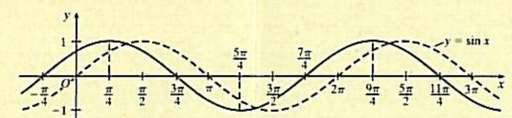
M (Trigonometry)

Trigonometric Functions

- Find $\sin\theta$, $\cos\theta$, and $\tan\theta$ when $\theta = 120°$.
  $\sin\theta = \frac{\sqrt{3}}{2}$
  $\cos\theta = -\frac{1}{2}$
  $\tan\theta = -\sqrt{3}$
- $\frac{2}{3}\pi$ *radians* $= 120°$

Trigonometric Graphs and Inequalities, Maxima and Minima

$y = \sin\left(x + \frac{\pi}{4}\right)$



The graph of $y = \sin\left(x + \frac{\pi}{4}\right)$ is a translation of the graph of $y = \sin x$, $-\frac{\pi}{4}$ unit(s) along the x-axis.

Addition Thereom

$$\cos 2x - \cos x = 0$$
$$(2\cos^2 x - 1) - \cos x = 0$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$
$$\therefore \cos x = -\frac{1}{2}, 1$$
$$\therefore x = 0, \frac{2}{3}\pi, \frac{4}{3}\pi$$

Coordinates of a Point, Equations of Straight Lines

- Find the equation of the line passing through point (2, 3) with slope 2.

  [Sol] Letting $y = 2x + b$ be the equation of the line,
  Since it passes through point (2, 3), substituting into the equation,
  $3 = 2 \cdot 2 + b$
  $b = -1$
  Therefore, $y = 2x - 1$

- Find the equation of the line passing through point (-3, 5) and perpendicular to $2x + 4y = 1$.

  [Sol] The slope of $2x + 4y = 1$ is $-\frac{1}{2}$.
  $y - 5 = 2(x+3) \qquad \therefore y = 2x + 11$

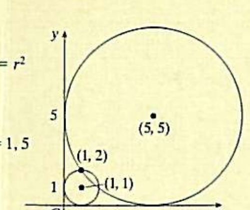
Equations of Circles and Tangent Lines of Circles

Find the equation of the two circles passing through (1, 2) touching both axes.

Letting the center be (r, r), the equation is: $(x-r)^2 + (y-r)^2 = r^2$

Since it passes through (1, 2),
From $(1-r)^2 + (2-r)^2 = r^2$, $r = 1, 5$

$$\begin{cases}(x-1)^2 + (y-1)^2 = 1 \\ (x-5)^2 + (y-5)^2 = 25\end{cases}$$



N (Calculus)

Loci, Quadratic Inequalities & Regions

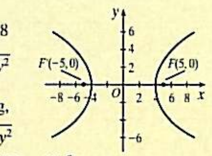
Given the fixed points $F(5, 0)$ and $F'(-5, 0)$, obtain the equation of the locus of point $P(x, y)$, such that $PF - PF' = 8$.

[Sol] Since $PF - PF' = 8$,

$$\sqrt{(x-5)^2 + y^2} - \sqrt{(x+5)^2 + y^2} = 8$$
$$\sqrt{(x-5)^2 + y^2} = 8 + \sqrt{(x+5)^2 + y^2}$$

Squaring both sides and simplifying,

$$-16 - 5x = 4\sqrt{(x+5)^2 + y^2}$$
$$256 + 160x + 25x^2 = 16x^2 + 160x + 400 + 16y^2$$
$$9x^2 - 16y^2 = 144$$
$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$



Arithmetic, Geometric, and Various Sequences

- $\sum_{n=1}^{3}(k-n) = (k-1) + (k-2) + (k-3) = 3k - 6$
- Evaluate $1 \cdot 4 + 4 \cdot 7 + 7 \cdot 10 + \cdots\cdots + (n^{th}\ term)$

  [Sol] $a_k = (3k-2)(3k+1)$
  $= 9k^2 - 3k - 2$

  $$\therefore \sum_{k=1}^{n} a_k = 9\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k - 2n$$
  $$= 9 \cdot \frac{n(n+1)(2n+1)}{6} - 3 \cdot \frac{n(n+1)}{2} - 2n$$
  $$= n(3n^2 + 3n - 2)$$

Infinite Sequences

- $\lim_{n \to \infty} \frac{2n^2 - 1}{3n^2 + 4} = \lim_{n \to \infty} \frac{2 - \frac{1}{n^2}}{3 + \frac{4}{n^2}} = \frac{2}{3}$
- $\lim_{n \to \infty} \frac{1 + 2 + \cdots + n}{n^2} = \lim_{n \to \infty} \frac{\frac{n(n+1)}{2}}{n^2} = \lim_{n \to \infty} \frac{n+1}{2n}$
  $= \lim_{n \to \infty} \frac{1 + \frac{1}{n}}{2} = \frac{1}{2}$

Infinite Series

State whether the given series converges or diverges.

$\frac{1}{1} + \frac{2}{3} + \frac{3}{5} + \frac{4}{7} + \cdots\cdots$

[Sol] $a_n = \frac{n}{2n-1} \qquad \therefore \lim_{n \to \infty} a_n = \frac{1}{2} \neq 0$

Therefore, the series diverges.

Limits of Functions and Continuity

$$\lim_{x \to 0} \frac{\sin x}{x + \sin x} = \lim_{x \to 0} \frac{\frac{\sin x}{x}}{1 + \frac{\sin x}{x}} = \frac{1}{2}$$

Differentiation

$$y = \left(\frac{2x-1}{x^3+7}\right)^{-2}$$
$$y' = -2\left(\frac{2x-1}{x^3+7}\right)^{-3} \cdot \left(\frac{2x-1}{x^3+7}\right)'$$
$$= -2\left(\frac{2x-1}{x^3+7}\right)^{-3} \cdot \frac{2(x^3+7) - (2x-1)(3x^2)}{(x^3+7)^2}$$
$$= -2\left(\frac{x^3+7}{2x-1}\right)^{3} \cdot \frac{-4x^3 + 3x^2 + 14}{(x^3+7)^2}$$
$$= \frac{2(x^3+7)(4x^3 - 3x^2 - 14)}{(2x-1)^3}$$

O (Calculus)

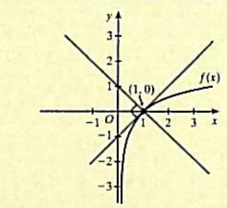
Advanced Differentiation

Find the tangent line and the normal line to $y = \ln x$ at point (1, 0).

[Sol] Letting $f(x) = \ln x$
$f'(x) = \frac{1}{x}$
$f'(1) = 1$

The equation of the tangent line is: $y = x - 1$

The equation of the normal line is: $y = -x + 1$



Applications of Differential Calculus

- Obtain the maximum and minimum values of $y = x - \cos x$. $(0 \leq x \leq 2\pi)$

  [Sol] $y' = 1 + \sin x \geq 0$

x	0	...	$\frac{3\pi}{2}$	...	2π
y'	+	+	0	+	+
y	−1	↗	$\frac{3\pi}{2}$	↗	$2\pi - 1$

From the table,
The maximum value is $2\pi - 1$, (at $x = 2\pi$).
The minimum value is -1, (at $x = 0$).

- Given the function $(2x-3)^2 = y^2 + x$, find $\frac{dy}{dx}$.

  [Sol] Differentiating both sides with respect to x,
  $4(2x-3) = 2y\frac{dy}{dx} + 1 \qquad \therefore \frac{dy}{dx} = \frac{8x-13}{2y}$

Indefinite Integrals

$\int \frac{e^x}{e^x + 1} dx$

[Sol] Letting $t = e^x$, $\quad e^x\, dx = dt$

$$\therefore \int \frac{e^x}{e^x+1} dx = \int \frac{1}{t+1} dt$$
$$= \ln|t+1| + C$$
$$= \ln(e^x + 1) + C$$

Definite Integrals

$$\int_0^{\frac{\pi}{2}} x \sin x\, dx$$
$$= -\Big[x \cos x\Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\, dx$$
$$= \Big[\sin x\Big]_0^{\frac{\pi}{2}}$$
$$= 1$$

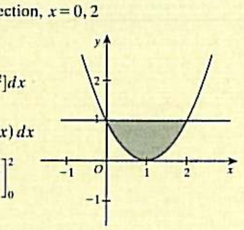
Applications of Integrals

Determine the volume generated when the region bounded by the curve $y = x^2 - 2x + 1$ and the line $y = 1$ is rotated around the x-axis.

[Sol] Finding the points of intersection, $x = 0, 2$

Letting V be the volume,

$$V = \pi\int_0^2 [(1)^2 - (x^2 - 2x + 1)^2]dx$$
$$= \pi\int_0^2 (-x^4 + 4x^3 - 6x^2 + 4x)\, dx$$
$$= \pi\left[-\frac{1}{5}x^5 + x^4 - 2x^3 + 2x^2\right]_0^2$$
$$= \frac{8\pi}{5}$$



Differential Equations

$y - x\frac{dy}{dx} = 1 \qquad (x = 1, y = 2)$

$$\frac{dy}{y-1} = \frac{dx}{x}$$
$$\int \frac{dy}{y-1} = \int \frac{dx}{x}$$
$$\ln|y-1| = \ln|x| + c$$
$$y - 1 = \pm x \cdot e^c$$
$$y = 1 + kx$$

From the initial condition, $k = 1$
Therefore, $y = 1 + x$